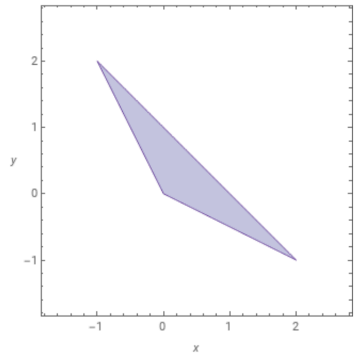
Tutorial Notes 2

1. Find the region represented by the following integral and calculate the integral:
$$\int_{-1}^{0}\int_{-2x}^{1-x} \mathrm{d}y\,\mathrm{d}x + \int_{0}^{2}\int_{-x/2}^{1-x} \mathrm{d}y\,\mathrm{d}x.$$

   Solutions:

   The region is as follows:

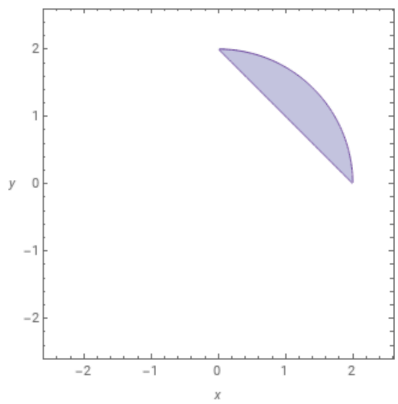


   From geometric meaning, the area is
$$\frac{1\times 1}{2} + \frac{1\times 2}{2} = \frac{3}{2}.$$
   From the integral, the area is
$$\int_{-1}^{0}(1+x)\,\mathrm{d}x + \int_{0}^{2}\left(1-\frac{x}{2}\right)\mathrm{d}x = \frac{1}{2} + 1 = \frac{3}{2}.$$

2. Find the area of the region $\Omega : 0 \le x \le 2,\ 2-x \le y \le \sqrt{4-x^2}$ using Fubini's theorem and geometry.

   Solutions:

   The region is as follows:



The area is

$$\begin{aligned}\int_0^2 \int_{2-x}^{\sqrt{4-x^2}} \mathrm{d}y\,\mathrm{d}x &= \int_0^2 [\sqrt{4-x^2}-(2-x)]\,\mathrm{d}x \\ &= \int_0^2 \sqrt{4-x^2}\,\mathrm{d}x - \int_0^2 (2-x)\,\mathrm{d}x \\ &= \int_0^{\pi/2} 2\cos\theta \cdot 2\cos\theta\,\mathrm{d}\theta + \left.\frac{(2-x)^2}{2}\right|_0^2 \\ &= \int_0^{\pi/2} 2(\cos 2\theta + 1)\,\mathrm{d}\theta - 2 \\ &= \pi - 2.\end{aligned}$$

From geometry, the area is

$$\frac{1}{4}\cdot\pi\cdot 2^2 - \frac{1}{2}\cdot 2^2 = \pi - 2.$$

3. Find the area of the region $\Omega : 1 \le r \le 1+\cos\theta$.

Solutions:

The area is

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta &= \int_{-\pi/2}^{\pi/2} \frac{(1+\cos\theta)^2-1}{2}\,\mathrm{d}\theta \\ &= \int_{-\pi/2}^{\pi/2} \frac{2\cos\theta+\cos^2\theta}{2}\,\mathrm{d}\theta \\ &= \int_{-\pi/2}^{\pi/2} \left(\cos\theta + \frac{1+\cos 2\theta}{4}\right)\mathrm{d}\theta \\ &= 2 + \frac{\pi}{4}.\end{aligned}$$

4. Find the area of one leaf of the rose $r = 12\cos 3\theta$.

Solutions:

The area is

$$\begin{aligned}\int_{-\pi/6}^{\pi/6}\int_0^{12\cos 3\theta} r\,\mathrm{d}r\,\mathrm{d}\theta &= \int_{-\pi/6}^{\pi/6} \frac{(12\cos 3\theta)^2}{2}\,\mathrm{d}\theta \\ &= \int_{-\pi/6}^{\pi/6} 72\cos^2 3\theta\,\mathrm{d}\theta \\ &= \int_{-\pi/6}^{\pi/6} 36(1+\cos 6\theta)\,\mathrm{d}\theta \\ &= 12\pi.\end{aligned}$$

5. For the region $\Omega : 0 \le r \le f(\theta)$, $\alpha \le \theta \le \beta$, where $f(\theta) \ge 0$ for $\alpha \le \theta \le \beta$, prove

that the area is

$$\int_{\alpha}^{\beta} \frac{f^2(\theta)}{2} \, \mathrm{d}\theta.$$

Solutions:

The area is

$$\int_{\alpha}^{\beta} \int_{0}^{f(\theta)} r \, \mathrm{d}r \, \mathrm{d}\theta = \int_{\alpha}^{\beta} \frac{f^2(\theta)}{2} \, \mathrm{d}\theta.$$